For small deviations from the initial conditions corresponding to a center, both in the case $I_1 < I_2$ (Fig. 1 b) as well as in the case $I_1 = I_2$ (Fig. 1 d), the representative point describes small circles around the center, i.e., the quantities ρ_i $(i = 1, 2, 3)$ perform small periodic oscillations around ρ_i^*. In the case $I_1 = I_2$ a periodic motion corresponds also to the singular point $(\rho_3 = I_1,\ \rho_3^{\cdot} = 0)$ for which only the one "fast" quasi-oscillator

$$\rho_3 = I_1, \quad \varphi_3(t) = -\beta_3 t + \varphi_{30}$$

"moves". However, the nature of this periodic motion is such that for the least change in the initial conditions the representative point (Fig. 1 d) starts to move along a cycle close to the separatrix, which corresponds to a "slow" pumping of energy between the oscillators.

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BIBLIOGRAPHY

1. Khazin, L.G. and Tsel'man, F.Kh., On the nonlinear interaction of resonant oscillators. Dokl. Akad. Nauk SSSR Vol.193, №2, 1970.
2. Tsel'man, F.Kh., On pumping transfer of energy between nonlinearly-coupled oscillators in third-order resonance. PMM Vol.34, №5, 1970.
3. Moser, J., Lectures on Hamiltonian Systems. Memoirs Amer. Math. Soc. №81, 1968.
4. Briuno, A.D., Normal form of differential equations. Dokl. Akad. Nauk SSSR Vol.157, №6, 1964.
5. Vitt, A. and Gorelik, G., Oscillations of an elastic pendulum as an example of the oscillations of two parametrically-coupled linear systems. Zh. Tekhn. Fiz. Vol.3, №s 2, 3, 1933.
6. Mettler, E., Kleine Schwingungen und Methode der säkularen Störungen. Z. Angew. Math. Mech. Vol.43, Special Issue, pp. T81-T85, 1963.

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ON THE RELATION BETWEEN RADIAL AND VERTICAL OSCILLATIONS OF PARTICLES IN CYCLOTRONS

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We consider the betatron oscillations of particles in cyclotrons with weak focusing. The equations of motion of the particles are described in the form of a fourth-order Liapunov system [1, 2]. On the basis of a transformation of Liapunov systems, proposed by the author [3, 4], the equations of motion are reduced to a second-order nonautonomous equation containing a small parameter. The vertical-radial oscillations of the particles are determined with the aid of the

small parameter method and the transition process from radial oscillations to vertical-radial ones is described.

The equations of betatron oscillations of particles in cyclotrons with weak focusing [8] may be written in the form (*)

$$\xi^{\cdot\cdot} + \xi = -{}^1\!/_2\,\beta\eta^2, \qquad \eta^{\cdot\cdot} + \alpha\eta = -\beta\xi\eta \tag{1}$$
$$\left(\xi = \frac{r - r_0}{r_0},\ \eta = \frac{z}{r_0};\quad \alpha = \frac{n}{1-n},\quad \beta = \frac{k}{\omega^2(1-n)}\right)$$

Here r and z are two of the cylindrical coordinates of a particle, the dot on top denotes differentiation with respect to a dimensionless time τ and

$$\tau = \omega\sqrt{1-n}\,t, \qquad \omega = \frac{eH(r_0)}{mc}$$
$$n = -\frac{r_0}{H(r_0)}H'(r_0) \qquad (0 < n < 1), \qquad k = -\frac{r_0}{H(r_0)}H''(r_0)$$

m and e are the mass and the charge of the particle, c is the velocity of light, $H(r)$ is the vertical component of the magnetic field strength vector, r_0 is the radius of the trajectory corresponding to a given energy of the particle. System (1) possesses an integral of particle energy, corresponding to the oscillatory part of the motion,

$$\xi^2 + \xi^{\cdot 2} + \alpha\eta^2 + \eta^{\cdot 2} + \beta\xi\eta^2 = \mu^2 \qquad (\mu > 0) \tag{2}$$

Before proceeding to the transformation of system (1) we note that it admits of the solution (the purely radial oscillations of the particles)

$$\eta \equiv 0, \qquad \xi = \mu\cos(\tau - \tau_0) \tag{3}$$

from integral (2) it follows that the constant μ^2 equals the square of the amplitude of the purely radial oscillations. To judge the stability of the latter, in [1] we set

$$\xi = \mu\cos(\tau - \tau_0) + x, \qquad \eta = y$$

and obtain equations in variations of the form

$$x^{\cdot\cdot} + x = 0, \qquad y^{\cdot\cdot} + [\alpha + \beta\mu\cos(\tau - \tau_0)]\,y = 0 \tag{4}$$

Hence it follows that the instability of the purely radial oscillations (3) is determined by the instability of the trivial solution of the second of Eqs. (4) – a Mathieu equation. On the (μn)-plane the instability regions abut the critical points n_l on the n-axis with tangent slope χ,

$$n_l = \frac{l^2}{4 + l^2}, \qquad \chi = \mp\frac{k}{2\omega^2}(1 - n_l) \qquad (l = 1, 2, \ldots)$$

Thus, in the first approximation, the instability regions (3) in the (μn)-plane are determined by the inequalities

$$\frac{l^2}{4 + l^2} - \frac{2k}{\omega^2(4 + l^2)}\mu + O(\mu^2) < n < \frac{l^2}{4 + l^2} + \frac{2k}{\omega^2(4 + l^2)}\mu + O(\mu^2)$$

for any positive integer l. Of greatest interest is the first instability regioı ($l = 1$) with the critical value $n_1 = {}^1\!/_5$.

*) Here in the first equation in (1) we have corrected a misprint which crept in in [8]

To seek periodic solutions of system (1), other than the purely radial oscillations, we transform it. Using the Liapunov substitution [1]

$$\xi = \rho \sin \vartheta, \qquad \xi = \rho \cos \vartheta, \qquad \eta = \rho \zeta \tag{5}$$

and integral (2) and following [3, 4], we arrive at the one equation (the prime denotes the derivative with respect to ϑ)

$$\zeta'' + \alpha\zeta = -\mu\beta\zeta \Big[\sqrt{1 + \alpha\zeta^2 + \zeta'^2} \sin \vartheta + $$
$$+ (1 + \alpha\zeta^2 + \zeta'^2)^{-1/2} \Big(\frac{1 - 4\alpha}{2} \zeta \sin \vartheta - {}^3/_2\, \zeta' \cos \vartheta \Big) \zeta \Big] + O(\mu^2) \tag{6}$$

Following Poincaré [6] we seek the periodic solution of Eq. (6) in series form

$$\zeta(\vartheta;\ \mu) = \zeta_0(\vartheta) + \mu\zeta_1(\vartheta) + \mu^2\zeta_2(\vartheta) + \ldots$$

and by substituting this series into Eq. (6) we obtain the equations for determining ζ_0 and ζ_1

$$\zeta_0'' + \alpha\zeta_0 = 0 \tag{7}$$

$$\zeta_1'' + \alpha\zeta_1 = -\beta\zeta_0 \Big[\sqrt{1 + \alpha\zeta_0^2 + \zeta_0'^2} \sin \vartheta + $$
$$+ (1 + \alpha\zeta_0^2 + \zeta_0'^2)^{-1/2} \Big(\frac{1 - 4\alpha}{2} \zeta_0 \sin \vartheta - {}^3/_2\, \zeta_0' \cos \vartheta \Big) \zeta_0 \Big] \tag{8}$$

Equation (7) possesses a family of $T(\alpha)$-periodic solutions

$$\zeta_0 = M_0 \cos \sqrt{\alpha}\, \vartheta + N_0 \sin \sqrt{\alpha}\, \vartheta \qquad (T(\alpha) = 2\pi\alpha^{-1/2}) \tag{9}$$

Solution (9) can be looked upon also as being $qT(\alpha)$-periodic, where q is any positive integer. Equation (6) depends explicitly on the independent variable ϑ, this dependency also can be treated as being $2p\pi$-periodic with any positive integer p. Therefore, solution (9) is the generating solution for a $2p\pi$-periodic solution of Eq. (6) if and only if

$$qT(\alpha) = 2p\pi, \ \text{i.e.,} \quad \alpha = \frac{q^2}{p^2} \quad \text{or} \quad n = \frac{q^2}{p^2 + q^2} \tag{10}$$

where q and p are any relatively prime numbers. Thus, Eq. (6) admits of periodic solutions with smallest period $2p\pi$ ($p = 1, 2, \ldots$) only for the α defined by formula (10) The set of values $\{n\}$ defined by formula (10) is everywhere dense on the interval $(0, 1)$ of variation of n, in other words, each value of $n \in (0, 1)$ either is defined by formula (10) or can be a representation of it to any degree of accuracy.

With due regard to (9) and (10), Eq. (8) is the equation for determining the first correction with respect to μ of the $2p\pi$-periodic solution of (6) ($p = 1, 2, \ldots$). The inhomogeneous part of this equation contains trigonometric functions with circular frequencies

a) $|p - q|/p$, b) $|p - 3q|/p$, c) $(p + q)/p$, $(p + 3q)/p$

Let us ascertain when one of these frequencies coincides with the circular frequency of the generating solution:

a) $(p - q)/p = q/p,\ p = 2,\ q = 1,\ \alpha = {}^1/_4,\ n = {}^1/_5$
b) $(p - 3q)/p = q/p,\ p = 4,\ q = 1,\ \alpha = {}^1/_{16},\ n = {}^1/_{17}$
c) $(3q - p)/p = q/p,\ p = 2,\ q = 1,\ \alpha = {}^1/_4,\ n = {}^1/_5$

In case (c) such a coincidence is impossible. In cases (a) and (b) Eq. (8) admits of $2p\pi$-periodic solutions for the indicated p not for all values of M_0 and N_0, but only for those for which the terms with $\sin(q\vartheta/p)$ and $\cos(q\vartheta/p)$ in its right-hand side are annulled. The equations for the generating amplitudes for $n = 1/5$

$$N_0(4 - 2M_0^2 + N_0^2) = 0, \qquad M_0(4 + M_0^2 - 2N_0^2) = 0$$

yield the nonzero solutions: $M_0 = \pm 2$, $N_0 = \pm 2$. From (9) we then obtain

$$\zeta_0 = \pm 2\sqrt{2}\cos(1/2\,\vartheta \mp 1/4\,\pi) \tag{11}$$

i.e., the only value of the generating amplitude equals $2\sqrt{2}$ for the four values of the generating initial phase.

In case $n = 1/17$ the equations for the generating amplitudes turn into identities. Therefore, for all the remaining values in (10), except $n = 1/5$, formula (9) supplies the family of generating solutions of Eq. (6) from two parameters.

In conclusion we dwell on $n = 1/5$, the smallest value of n for which the purely radial oscillations (3) are unstable with an arbitrarily small amplitude μ. Following [3,4] from formulas (5), (11) we have

$$\rho = 1/3\sqrt{3}\mu + O(\mu^2), \quad \vartheta^{\cdot} = 1 + 2/3\sqrt{3}\beta\mu(1 \pm \sin\vartheta)\sin\vartheta + O(\mu^2)$$

Hence for the period of the vertical-radial oscillations (the desired periodic solution) we obtain

$$T = \frac{\sqrt{5}}{2\omega}\int_0^{4\pi}[1 + 2/3\sqrt{3}\beta\mu(1 \pm \sin\vartheta)\sin\vartheta + O(\mu^2)]^{-1}d\vartheta =$$

$$= \frac{2\sqrt{5}\pi}{\omega}\left[1 \mp 5/12\sqrt{3}\frac{k}{\omega^2}\mu + O(\mu^2)\right] \tag{12}$$

for the law of motion

$$\xi = \rho\sin\vartheta = 1/3\sqrt{3}\mu\sin(2/5\sqrt{5}\omega t) + O(\mu^2)$$
$$\eta = \rho\zeta = \pm 2/3\sqrt{6}\mu\cos(1/5\sqrt{5}\omega t \mp 1/4\pi) + O(\mu^2) \tag{13}$$

The magnitude of μ is determined by the initial value of the derived energy (2) of the oscillations.

Let us go on to describe the pumping of energy when $n = 1/5$ ($\alpha = 1/4$), i.e., the transition process from unstable purely radial oscillations (3) to the vertical-radial oscillations (13). The Van der Pol substitution [7, 5]

$$\zeta = a\cos(1/2\vartheta + \varphi), \qquad \zeta' = -1/2\, a\sin(1/2\vartheta + \varphi) \tag{14}$$

and a subsequent averaging over the explicitly-occurring independent variable ϑ lead Eq. (6), for $\alpha = 1/4$ to truncated Can der Pol equations in the slowly-varying variables a and φ

$$a' = 1/4\,\mu\beta a\sqrt{4 + a^2}\cos 2\varphi + O(\mu^2), \quad \varphi' = -1/8\,\mu\beta\frac{8 - a^2}{\sqrt{4 + a^2}}\sin 2\varphi + O(\mu^2) \tag{15}$$

An exact integration of this system leads to quadratures which are difficult to handle. We restrict ourselves to an approximate integration of it. From the second of Eqs. (15) it follows that when $|a| \leqslant 2\sqrt{2}$ we have $\varphi' \leqslant 0$ and thus $|\varphi_0| > |\varphi| \geqslant 0$. By virtue of (14) since we have $|a| \leqslant 2\sqrt{2}$ when $|\zeta| \leqslant 2\sqrt{2}$, where, we recall, $2\sqrt{2}$ is the amplitude of the generating solution (11) during the whole of the transition interval being considered we can set $\cos 2\varphi \approx 1$ if φ_0 is sufficiently small. Then the first of Eqs. (15) gives us for $a_0 = a(0) > 0$

$$\int_{a_0}^{a} \frac{da}{a\sqrt{4+a^2}} = \frac{1}{4}\mu\beta\vartheta$$

Hence we obtain an approximate law of variation of the Van der Pol amplitude a

$$a = \frac{4b_0 \exp(^1/_2\,\beta\mu\vartheta)}{1 - b_0^2 \exp(\beta\mu\vartheta)} \qquad \left(b_0 = \frac{1}{a_0}\,[\sqrt{4+a_0^2} - 2]\right) \tag{16}$$

The first of formulas (14) now describes the transition process. Let us ascertain the time of transition from the purely radial oscillations (3) to the vertical-radial ones (13). By setting $a = 2\sqrt{2}$ in (16), for the corresponding value of $\vartheta = \theta$ we have

$$\theta = \frac{2}{\beta\mu} \ln\left(\frac{\sqrt{3}-1}{\sqrt{2}}\, \frac{a_0}{\sqrt{4+a_0^2} - 2}\right) \tag{17}$$

We note that the transition process takes place over a time interval whose duration is of the order $O\,(1\,/\,\mu)$, which corresponds to the algorithm of asymptotic integration in the averaging method [9]. Because φ_0 is small we have $a_0 \approx \zeta\,(0)$ and for small a_0 formula (17) takes the form

$$\theta = \frac{2}{\beta\mu} \ln \frac{2\sqrt{2}(\sqrt{3}-1)}{\zeta\,(0)} \tag{18}$$

Let us express $\zeta\,(0)$ in terms of the initial values of the original variables,

$$\zeta = \frac{\eta}{\rho}, \quad \rho = \sqrt{\xi^2 + \xi'^2}, \quad \zeta\,(0) = \eta\,(0)\left[\xi\,(0)^2 + {}^5/_4\,\omega^{-2}\left(\frac{d\xi}{dt}\right)_0^2\right]^{-1/2}$$

Analogously to (12) we determine the original time for the transition process from the purely radial oscillations (3) to the vertical-radial ones (13) for $n = {}^1/_5$,

$$T = \frac{\sqrt{5}}{2\omega}\left[\left(1 \mp {}^5/_{12}\,\sqrt{3}\,\frac{k}{\omega^2}\,\mu\right)\theta + O\,(\mu)\right]$$

where the choice of sign is determined by the choice within the brackets in (13), while the magnitude of μ is determined by the initial value of the total energy (2) of the oscillations.

BIBLIOGRAPHY

1. Liapunov, A. M., General Problem of the Stability of Motion. Khar kov, 1892.
2. Malkin, I. G., The Methods of Liapunov and Poincaré in the Theory of Nonlinear Oscillations. Moscow-Leningrad, Gostekhizdat, 1949.
3. Starzhinskii, V. M., Vereinigungen der Methoden von Ljapunow und Poincaré in der Theorie nichtlinearen Schwingungen. Berlin, Kl. Math. Phys. und Techn., №1, 1965.
4. Starzhinskii, V. M., On one variant of a method for determining periodic solutions. Inzh. Zh. MTT, №6, 1968.
5. Van der Pol, B., Nonlinear Theory of Electric Oscillations. Moscow, Sviaz izdat, 1935.
6. Poincaré, H., Les Méthodes Nouvelles de la Mécanique Céleste. Tomes 1-3. Paris, 1892-1899.

7. Mitropol'skii, Iu. A., Lectures on the Averaging Method in Nonlinear Mechanics. Kiev, "Naukova Dumka", 1966.
8. Cohen, B., Cyclotrons and phasotrons. In: Accelerators. Gosatomizdat, 1962.
9. Moiseev, N. N., Asymptotic Methods of Nonlinear Mechanics. Moscow, "Nauka" 1969.

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MOTION OF A GAS BEHIND A PLANE DETONATION WAVE ORTHOGONAL TO THE FREE SURFACE

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The problem of detonation of one quarter of a space filled with explosive and initiated on one of the faces is examined. The finding of the solution in the perturbed region is reduced to the solution of Goursat's problem for a quasi-linear differential equation of second order with two independent variables. This problem is solved by the numerical method of characteristics. An examination of singular points is presented. The solution in the perturbed region and the form of the free surface are obtained.

The problem of gas motion behind an expanding detonation wave in a space with a conical cutout was examined in papers [1, 2].

1. Let us examine the infinite region

$$x_1 > 0, \qquad x_2 < 0 \quad \text{for} \quad t < 0 \tag{1.1}$$

filled with immovable explosive of constant density ρ_0. The pressure p in the entire space is equal to zero.

It will be assumed that the products of explosion are described by the following equation of state

$$p = \gamma\rho^\gamma, \qquad \gamma > 1 \qquad (\rho_0 = \gamma/(\gamma+1)) \tag{1.2}$$

At the instant of time $t = 0$ the explosive is initiated on the surface $x_1 = 0$. A plane normal detonation wave, which is orthogonal to the free surface $x_2 = 0$ propagates with the constant velocity $D = \gamma + 1$ from the plane of initiation $x_1 = 0$

2. For $t > 0$ the motion is self-similar with the following independent variables

$$\xi_1 = x_1/t, \qquad \xi_2 = x_2/t \tag{2.1}$$

The straight line $\xi_1 = D$ corresponds to the front of the detonation wave. Behind the wave front the gasdynamic parameters assume the following values

$$u_1 = 1, \quad u_2 = 0, \quad \rho = 1, \quad c = \gamma \tag{2.2}$$

where u_1 and u_2 are components of the velocity vector, c is the speed of sound. Far from the straight line $\xi_2 = 0$ in the region

$$-D/(\gamma - 1) < \xi_1 < D$$